Fast Training of Implicit Networks with Applications in Inverse Problems

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What are Inverse Problems?

Inverse problems consist of recovering a signal x^* (e.g. an image, a parameter of a PDE, etc.) from indirect, noisy measurements d. This measurement process is usually modeled as an operator $\mathcal{A}$, satisfying the following:

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Our task deals with image deblurring, i.e.,

- $d \in \mathbb{R}^{n \times n}$: blurred image with noise
- $x^* \in \mathbb{R}^{n \times n}$: original image
- $\varepsilon \in \mathbb{R}^{n \times n}$: random noise (unknown) in $\mathbb{R}^{n \times n}$

From a Classical Approach

Direct Inverse:

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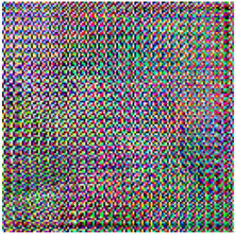
Original Image

Blurred Noisy Image

Apply Inverse



Classical Approach Cont.

Optimization: Formulate an optimization problem as follows:

$$x^* = \underset{x \in \mathbb{R}^{n \times n}}{\arg\min} \ \frac{1}{2}||\mathcal{A}x - d||_{L^2}^2 + \lambda R(x)$$

where $R(x)$ is chosen based on prior knowledge of your data, $\lambda > 0$ is a tunable parameter.

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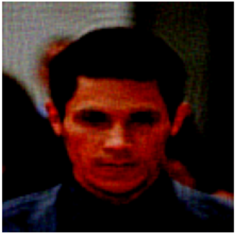
E.g. Use gradient descent where $R(x) = \frac{\lambda}{2}||x||_{L^2}^2$:

Original Image | Blurred Noisy Image | Apply Gradient Descent



Implicit Deep Learning

- Use dataset $\{(d_i, x_i^*)\}_{i=1}^m$ and physics (namely $\mathcal{A}$)

[5] Davis Gilton, Gregory Ongie, and Rebecca Willett. "Deep equilibrium architectures for inverse problems in imaging." IEEE Transactions on Computational Imaging 7 (2021): 1123-1133.

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- Mimic gradient descent [5], but replace $\lambda \nabla_x R$ with a trainable network S_Θ: $\forall i$ and $0 \leq k \leq K-1$,

$$x_i^{k+1} = \underbrace{x_i^k - \eta \left(\nabla_x ||\mathcal{A} x_i^k - d_i||_{L^2}^2 + S_\Theta(x_i^k) \right)}_{:= T_\Theta(x_i^k)}$$

where:

- $\eta > 0$ is step size
- $T_\Theta(\cdot)$ is a layer of our neural network $\mathcal{N}_\Theta(\cdot)$
- K is the number of layers

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- Problems: memory, choice of K

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- - Convergent if $T_\Theta(\cdot)$ is a contraction mapping with Lipschitz constant $\gamma \in [0, 1)$, i.e,
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  Then, by *Banach fixed-point theorem*, there exists $y^* \in \mathbb{R}^{n^2}$ s.t. $T_\Theta(y^*) = y^*$

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$$\implies \left(I - \frac{dT_\Theta(x^*)}{dx^*}\right)\frac{dx^*}{d\Theta} = \frac{\partial T_\Theta(x^*)}{\partial\Theta} \tag{1}$$

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where $\alpha > 0$ is the learning rate.

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Potential problem: solving (1) is highly nontrivial

Jacobian-Free Backpropagation (JFB)

- Goal: alleviate memory requirement and avoid high computational cost.

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- JFB approximates the gradient: $p_\Theta = \frac{d\ell}{dx^*}\frac{\partial T_\Theta(x^*)}{\partial\Theta}$ which is a **descent direction** for ℓ if the following conditions hold (next slide) [6]:

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JFB Conditions

If:

i. T_Θ is contraction mapping with Lipschitz constant γ

ii. T_Θ is continuously differentiable w.r.t. Θ

iii. $M := \frac{\partial T_\Theta}{\partial \Theta}$ has full column rank

iv. M is well-conditioned, i.e., $\kappa(M^T M) < \frac{1}{\gamma}$

Then

$$p_\Theta = \frac{d\ell}{dx^*} \frac{\partial T_\Theta}{\partial \Theta}$$

is a **descent direction** for loss function ℓ.

Numerical Experiments

- Dataset: CelebA [7] (annotated celebrity faces)

[7] Liu, Ziwei, et al. "Deep learning face attributes in the wild." Proceedings of the IEEE international conference on computer vision. 2015.

Numerical Experiments

- Generate blurred noisy images:

original image

blurred noisy image

PSNR = 21.57, SSIM=0.80

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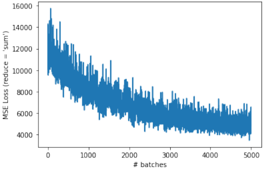
- Generate blurred noisy images:

original image | blurred noisy image

PSNR = 21.57, SSIM=0.80

- Train with JFB
- Preliminary results:

Loss v.s. number of SGD iterations | Reconstructed image



PSNR = 25.69, SSIM=0.86

Future Work

- Train models using different learned optimization algorithms, e.g. *Proximal Gradient Descent* and *Alternating Directions Method of Multipliers (ADMM)*
- Experiment with fastMRI data [8]
- Compare training speeds and accuracy with Jacobian-based algorithms

[8] Zbontar, Jure, et al. "fastMRI: An open dataset and benchmarks for accelerated MRI." arXiv preprint arXiv:1811.08839 (2018).

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Thank you!

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Banach Fixed-Point Theorem

We demonstrate it in $\mathbb{R}^d$:
Suppose $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a contraction map with Lipschitz constant $\gamma \in [0, 1)$.
$\forall x_0 \in \mathbb{R}^d$, iterate as follows:

$$\begin{aligned} x_1 &= T(x_0) \\ x_2 &= T(x_1) \\ &\vdots \\ x_{i+1} &= T(x_i) \\ &\vdots \end{aligned}$$

Then we obtain a sequence $\{x_m\}_{m \in \mathbb{N}}$

Banach Fixed-Point Theorem

Observe that

$$
\begin{aligned}
||x_2 - x_1|| &= ||T(x_1) - T(x_0)|| \leq \gamma ||x_0 - x_1|| \\
||x_3 - x_2|| &= ||T(x_2) - T(x_1)|| \leq \gamma ||x_2 - x_1|| \leq \gamma^2 ||x_0 - x_1|| \\
&\vdots \\
||x_{i+1} - x_i|| &\leq \gamma^i ||x_0 - x_1|| \\
&\vdots
\end{aligned}
$$

So $\lim_{m\to\infty} ||x_{m+1} - x_m|| \leq \lim_{m\to\infty} \gamma^m ||x_0 - x_1|| = 0$
We also know that $0 \leq \lim_{m\to\infty} ||x_{m+1} - x_m||$.
$\implies \lim_{m\to\infty} ||x_{m+1} - x_m|| = 0$ by *Squeeze Theorem*

Banach Fixed-Point Theorem

By *Triangular Inequality*,

$$
\begin{aligned}
||x_{m+k} - x_m|| &\leq ||x_m - x_{m+1}|| + ||x_{m+1} - xm + k|| \\
&\leq ||x_m - x_{m+1}|| + (||x_{m+1} - x_{m+2}|| + ||x_{m+2} - x_{m+k}||) \\
&\vdots \\
&\leq ||x_m - x_{m+1}|| + ||x_{m+1} - x_{m+2}|| + \cdots \\
&+ ||x_{m+k-2} - x_{m+k-1}|| + ||x_{m+k-1} - x_{m+k}||
\end{aligned}
$$

By *Squeeze Theorem* again, $\{x_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence, which is equivalent to $\lim\limits_{m\to\infty} x_m = x^*$ exists.
$\implies x^* = T(x^*)$ is a fixed point. $\square$

Proximal Gradient Descent

With a function $h(\cdot)$, we can define a proximal operator

$$\mathrm{prox}_h(x) = \arg\min_u \frac{1}{2}||u - x||^2_{L^2} + h(u)$$

Then the updating rule becomes:

$$x^{k+1} = \mathrm{prox}_{h,\eta}\left(x^k - \eta\nabla_x||\mathcal{A}x^k - d||^2_{L^2}\right)$$

We can replace this prox_h with a trainable network $R_\Theta(\cdot)$:

$$x^{k+1} = R_\Theta\left(x^k - \eta\nabla_x||\mathcal{A}x^k - d||^2_{L^2}\right)$$